

# Polymorphism, Subtyping, and Type Inference in MLsub

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# The select function

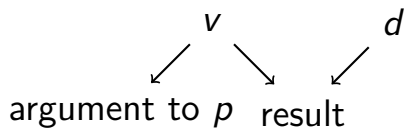
`select p v d = if (p v) then v else d`

In ML, select has type scheme

$$\forall \alpha. (\alpha \rightarrow \text{bool}) \rightarrow \alpha \rightarrow \alpha \rightarrow \alpha$$

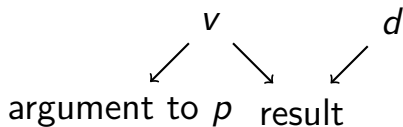
# Data flow in select

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select  $p$   $v$   $d = \text{if } (p \ v) \text{ then } v \text{ else } d$



In MLsub, select has this type scheme:

$$\forall \alpha, \beta. (\alpha \rightarrow \text{bool}) \rightarrow \alpha \rightarrow \beta \rightarrow (\alpha \sqcup \beta)$$

$$\Gamma \vdash e : \tau$$

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# Expressions of MLsub

We have functions

$x$        $\lambda x.e$        $e_1 e_2$

... and records

$\{l_1 = e_1, \dots, l_n = e_n\}$        $e.l$

... and booleans

true      false      if  $e_1$  then  $e_2$  else  $e_3$

... and let

$\hat{x}$       let  $\hat{x} = e_1$  in  $e_2$

$\Gamma \vdash e : \tau$



# Typing rules of MLsub

MLsub is

ML +

$$(\text{SUB}) \quad \frac{\Gamma \vdash e : \tau_1}{\Gamma \vdash e : \tau_2} \quad \tau_1 \leq \tau_2$$

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# Constructing Types

The standard definition of types looks like:

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(ignoring records and booleans for now)

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with a subtyping relation like:

$$\frac{}{\perp \leq \tau} \quad \frac{}{\tau \leq \top} \quad \frac{\tau'_1 \leq \tau_1 \quad \tau_2 \leq \tau'_2}{\tau_1 \rightarrow \tau_2 \leq \tau'_1 \rightarrow \tau'_2}$$

# Lattices

These types form a lattice:

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$$\frac{e_1 : \tau_1 \quad e_2 : \tau_2}{\text{if rand } () \text{ then } e_1 \text{ else } e_2 : \tau_1 \sqcup \tau_2}$$

# Bizzarely difficult questions

Is this true, for all  $\alpha$ ?

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Yes, it turns out, by **case analysis** on  $\alpha$ .

And *only* by case analysis.

# Extensibility

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Now we have a counterexample:

$$\alpha = (\top \overset{\circ}{\rightarrow} \perp) \overset{\circ}{\rightarrow} \perp$$

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Two techniques give us an extensible system:

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*gets rid of case analysis*

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- ▶ Add explicit type variables as indeterminates  
*gets rid of case analysis*
- ▶ Require a distributive lattice  
*gets rid of vacuous reasoning*

# Combining types

How to combine different types into a single system?

$$\tau ::= \text{bool} \mid \tau_1 \rightarrow \tau_2 \mid \{l_1 : \tau_1, \dots, l_n : \tau_n\}$$



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We should read ‘|’ as **coproduct**

# Concrete syntax

Build an actual syntax for types, by writing down all the operations on types:

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then quotient by the equations of distributive lattices, and the subtyping order.

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- We end up with all the standard types
- ... with the same subtyping order
- ... but we identify fewer of the weird types

$$\{\text{foo} : \text{bool}\} \sqcap (\top \rightarrow \top) \not\leq \text{bool}$$

$\Gamma \vdash e : \tau$

# Principality in ML

Intuitively,

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but it's a bit more complicated than that:

*For any  $e$  typeable under  $\Gamma$ , there's a  $\tau$  **and a substitution**  $\sigma$  such that every possible typing of  $e$  under  $\Gamma$  is a substitution instance of  $\sigma\Gamma, \tau$ .*

# Reformulating the typing rules

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Instead, split  $\Gamma$ :

- ▶  $\Delta$  maps  $\lambda$ -bound  $x$  to a type  $\tau$
- ▶  $\Pi$  maps let-bound  $\hat{x}$  to a *typing schemes*  $[\Delta]_{\tau}$

$$\Pi \Vdash e : [\Delta]_{\tau}$$

question      answer

$\overbrace{\prod \Vdash e} : \overbrace{[\Delta]_{\mathcal{T}}}$

# Subsumption

Define  $\leq^{\forall}$  as the least relation closed under:

- ▶ *Instatiation*, replacing type variables with types
- ▶ *Subtyping*, replacing types with supertypes

# Principality in MLsub

A *principal typing scheme* for  $e$  under  $\Pi$  is a  $[\Delta]\tau$  that subsumes any other.

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These are *equivalent* ( $\equiv^\forall$ ): subsume each other

# Input and output types

$\tau \sqcup \tau'$ : produces a value which is a  $\tau$  or a  $\tau'$

$\tau \sqcap \tau'$ : requires a value which is a  $\tau$  and a  $\tau'$

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Divide types into

- ▶ *output types*  $\tau^+$
- ▶ *input types*  $\tau^-$

# Polar types

$$\tau^+ ::= \text{bool} \mid \tau_1^- \rightarrow \tau_2^+ \mid \{\ell_1 : \tau_1^+, \dots, \ell_n : \tau_n^+\} \mid \\ \alpha \mid \tau_1^+ \sqcup \tau_2^+ \mid \perp \mid \mu\alpha.\tau^+$$

$$\tau^- ::= \text{bool} \mid \tau_1^+ \rightarrow \tau_2^- \mid \{\ell_1 : \tau_1^-, \dots, \ell_n : \tau_n^-\} \mid \\ \alpha \mid \tau_1^- \sqcap \tau_2^- \mid \top \mid \mu\alpha.\tau^-$$

# Cases of unification

In HM inference, unification happens in three situations:

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*Introduce*  $\sqcup$

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- ▶ Using the output of one expression as input to another

$\tau^+ \leq \tau^-$  *constraint*



# Eliminating variables, ML-style

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The substitution  $[\tau/\alpha]$  **solves** the constraint  $\alpha = \tau$

# “solves?”

What does it mean to **solve** a constraint?

1.  $[\tau/\alpha]$  trivialises the constraint  $\alpha = \tau$   
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(it is a *most general unifier*)
2. For any type  $\tau'$ , the following sets agree:  
the instances of  $\tau'$ , subject to  $\alpha = \tau$   
the instances of  $[\tau/\alpha]\tau'$

## Definition 2, now with subtyping

Suppose we have an identity function, which uses its argument as a  $\tau^-$ .

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The *bisubstitution*  $[\alpha \sqcap \tau^- / \alpha^-]$  solves  $\alpha \leq \tau^-$

# Decomposing constraints

We only need to decompose constraints of the form  $\tau^+ \leq \tau^-$ .

$$\tau_1 \sqcup \tau_2 \leq \tau_3 \quad \equiv \quad \tau_1 \leq \tau_3, \tau_2 \leq \tau_3$$

$$\tau_1 \leq \tau_2 \sqcap \tau_3 \quad \equiv \quad \tau_1 \leq \tau_2, \tau_1 \leq \tau_3$$

Thanks to the input/output type distinction, the hard cases of  $\tau_1 \sqcap \tau_2 \leq \tau_3$  and  $\tau_1 \leq \tau_2 \sqcup \tau_3$  can never come up.

# Combining solutions

We solve a system of multiple constraints  $C_1, C_2$  by:

- ▶ Solving  $C_1$ , giving a bisubstitution  $\xi$
- ▶ Applying that to  $C_2$
- ▶ Solving  $\xi C_2$ , giving a bisubstitution  $\zeta$

Then  $\xi \circ \zeta$  solves the system  $C_1, C_2$ .

# Putting it all together

$\text{biunify}(C)$  takes a set of constraints  $C$ , and produces a bisubstitution solving them.

$$\text{biunify}(\emptyset) = []$$

$$\text{biunify}(\alpha \leq \alpha, C) = \text{biunify}(C)$$

$$\text{biunify}(\alpha \leq \tau, C) = \text{biunify}(\theta_{\alpha \leq \tau} H; \theta_{\alpha \leq \tau} C) \circ \theta_{\alpha \leq \tau}$$

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Replace the  $\leq$  with  $=$  and we have Martelli and Montanari's unification algorithm.

# Summary

MLsub infers types by walking the syntax of the program, but must deal with subtyping constraints rather than just equalities. Thanks to:

- ▶ algebraically well-behaved types
- ▶ polar types, restricting occurrences of  $\sqcup$  and  $\sqcap$
- ▶ a careful definition of “solves”

the biunify algorithm can always handle these constraints, producing a principal type.

Questions?

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# Mutable references

References are generally considered “invariant”.  
Instead, consider `ref` a two-argument constructor

$$(\alpha, \beta) \text{ ref}$$

with operations:

$$\text{make} : (\alpha, \alpha) \text{ ref}$$
$$\text{get} : (\perp, \beta) \text{ ref} \rightarrow \beta$$
$$\text{set} : (\alpha, \top) \text{ ref} \rightarrow \alpha \rightarrow \text{unit}$$